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SOME ANNULAR DISC INCLUSION PROBLEMS IN ELASTICITY

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Abstract—This paper examines the problems related to the displacement and rotation of a rigid annular disc inclusion which is embedded in bonded contact with an isotropic elastic infinite space. The analysis of the inclusion problems can be reduced to the solution of sets of triple integral equations. These equations are solved in an approximate fashion to yield the rotational and translational stiffnesses for the embedded annular disc inclusion.

1. INTRODUCTION

The class of problems related to the behaviour of flexible or rigid disc shaped inclusions embedded in elastic media is of some interest to the study of multiphase elastic materials. The studies by Collins[1] and Keer[2] examine the problems of a rigid penny shaped inclusion embedded in bonded contact with an isotropic elastic solid. These studies were subsequently extended by Kassir and Sih[3] to include elliptical disc shaped rigid inclusions. The articles by Selvadurai[4–12] examine the problems related to elliptical or penny-shaped inclusions embedded in isotropic and transversely isotropic elastic media. Recent reviews of the subject of inclusions and inhomogeneities embedded in elastic media are given by Mura[13], Willis[14] and Walpole[15].

This paper examines a series of axisymmetric and asymmetric problems related to an annular rigid disc inclusion embedded in bonded contact with an isotropic elastic medium. The torsionless axisymmetric deformations are induced by the rigid body translation of the inclusion about the z-axis (where (r, θ, z) refers to the cylindrical polar coordinate system; Fig. 1). The asymmetric behaviour is induced by the rotation of the annular inclusion about the y-axis or the translation of the inclusion in the x-direction. The rotationally symmetric deformations are induced by the torsion of the annular disc inclusion about the z-axis. By virtue of the symmetrical geometry of the annular inclusion these problems examine, completely, the generalized displacements of the inclusion. The analysis of disc inclusion problems can be approached by employing a variety of analytical techniques. In the case of solid disc shaped inclusions, the solution to a particular problem can be obtained as a limiting case of the relevant problem for an oblate spheroidal inclusion. The spheroidal inclusion problem can be examined by making use of direct spheroidal harmonic function techniques[16–18] or singularity methods[19]. Alternatively the disc inclusion problem can be formulated as a mixed boundary value problem related to a half-space region. Owing to the topology of the annular disc inclusion it is not possible to employ methods based on spheroidal function techniques. Consequently the annular disc inclusion problems discussed previously should be formulated as mixed boundary value problems referred to a halfspace region. The use of such procedures is facilitated by the antisymmetry or symmetry that the problems exhibit about the plane $z = 0$. A Hankel transform development of these mixed boundary value problems yields sets of triple integral equations. The method proposed by Williams[20] is employed to derive approximate solutions to the systems of triple integral equations associated with the annular inclusion problems. The results of primary interest to engineering applications, namely, the axial, rotational and translational stiffnesses of the embedded annular disc inclusion are evaluated in series form in terms of a small non-dimensional geometric parameter. This parameter corresponds to the ratio of the inner to the outer radius of the annulus.

The class of problems related to the behaviour of flexible or rigid disc shaped inclusions embedded in elastic media is of some interest to geomechanics and to the study of multiphase

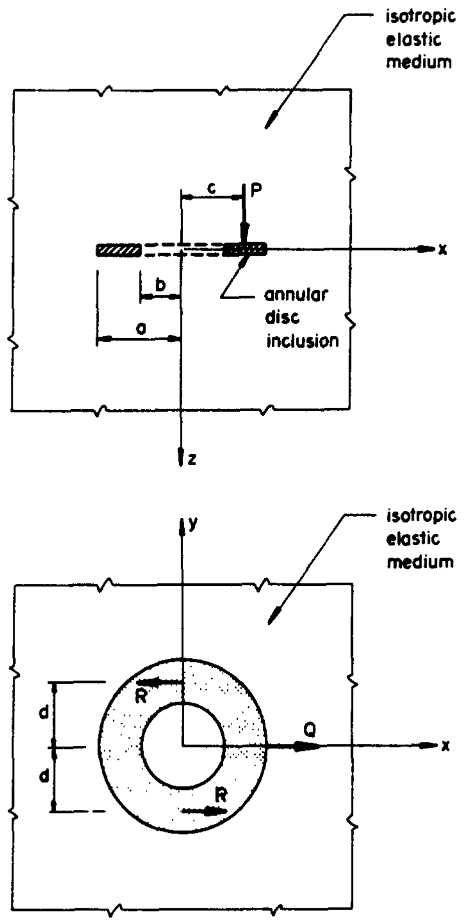

Fig. 1. Geometry of the annular disc inclusion and the resultant forces.

elastic materials. In geomechanical applications the rigid disc shaped inclusion represents the behaviour of an earth or a rock anchor which is created by the hydraulic fracture of the earth or rock mass. The disc shaped region represents the resinous or cementing material which is used to transfer the anchoring loads to the geologic medium. Alternatively the annular rigid inclusion represents, approximately, a foundation which is deeply embedded in a soil mass. In the context of composite materials dispersed disc inclusions or inhomogeneities are used to strengthen non-metallic or metallic matrices or increase the overall stiffness of a composite. The modes of deformation of the inclusion considered in this paper can be induced by the interaction of the composite and an electromagnetic field.

2. BASIC EQUATIONS

In connection with the solution of the axisymmetric and asymmetric problems related to the embedded annular inclusion it is convenient to employ a formulation based on the strain potential approach of Love[21] and its extension to asymmetric problems proposed by Muki[22]. It can be shown that these representations are specific reductions of the general method of analysis of the classical equations of elasticity in terms of the Boussinesq–Somigliana–Galerkin stress function[23, 24]. Proofs of the completeness of these representations are given by Truesdell[25] and Gurtin[23]. Also the uniqueness of solution derived from these potentials can be established by appeal to Kirchhoff's uniqueness theorem.

Briefly, the solution of the displacement equations of equilibrium, for an elastic medium free of body forces, can be represented in terms of a biharmonic function $\Phi(r, \theta, z)$ and a harmonic function $\Psi(r, \theta, z)$, i.e.:

$$\nabla^4\Phi(r, \theta, z) = 0; \nabla^2\Psi(r, \theta, z) = 0 \tag{1}$$

where

$$\nabla^4 = \nabla^2\nabla^2$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is Laplace's operator referred to the cylindrical polar coordinate system.

The components of the displacement vector $\mathbf{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ referred to the cylindrical polar coordinate system can be expressed in terms of the derivatives of Φ and Ψ. We have

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r\partial z} + \frac{2}{r}\frac{\partial\Psi}{\partial\theta} \tag{3a}$$

$$2Gu_\theta = -\frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial z} - 2\frac{\partial\Psi}{\partial r} \tag{3b}$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{3c}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio respectively. Similarly, the components of the stress tensor $\boldsymbol{\sigma}$ are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\Phi + \frac{\partial}{\partial\theta}\left(\frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right)\Psi \tag{4a}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\Phi - \frac{\partial}{\partial\theta}\left(\frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right)\Psi \tag{4b}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Phi \tag{4c}$$

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Phi - \frac{\partial^2\Psi}{\partial r\partial z} \tag{4d}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Phi + \frac{1}{r}\frac{\partial^2\Psi}{\partial\theta\partial z} \tag{4e}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}\left[\frac{1}{r} - \frac{\partial}{\partial r}\right]\Phi - \left[2\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial z^2}\right]\Psi. \tag{4f}$$

It may be noted that for axial symmetry $\Phi = \Phi(r, z)$ and $\Psi = 0$; thus the results (3) and (4) for the displacements and stresses reduce to the representation in terms of Love's strain potential.

3. THE ANNULAR DISC INCLUSION PROBLEM

We consider the problem of an annular rigid disc inclusion of external radius "a" and internal radius "b" which is embedded in bonded contact with an elastic infinite space (Fig. 1). The annular inclusion is subjected to a system of forces and couples which causes it to undergo

(i) a rigid body translation δ in the +ve z-direction, (ii) a rigid body rotation Ω about the y-axis, (iii) a rigid body rotation ω about z-axis and (iv) a rotation free lateral translation Δ in the x-direction. By examining the mode of deformation for each specific problem it can be shown that the embedded inclusion imposes certain symmetry properties in the displacements and stresses, in the infinite space, about the plane $z = 0$. We may therefore restrict the analysis to a single halfspace region in which the plane $z = 0^+$ is subjected to appropriate mixed boundary conditions. For convenience, we shall focus the attention on the halfspace region $z \geq 0$. The positive superscript denotes this case. The relevant boundary conditions are summarized.

(i) *For the rigid body translation in the z-direction*

$$u_r(r, 0^+) = 0; \; r \geq 0 \tag{5a}$$

$$u_z(r, 0^+) = \delta; \; b \leq r \leq a \tag{5b}$$

$$\sigma_{zz}(r, 0^+) = 0; \; a < r < \infty \tag{5c}$$

$$\sigma_{zz}(r, 0^+) = 0; \; 0 < r < b. \tag{5d}$$

(ii) *For the rigid body rotation about the y-axis*

$$u_r(r, \theta, 0^+) = 0; \; r \geq 0 \tag{6a}$$

$$u_\theta(r, \theta, 0^+) = 0; \; r \geq 0 \tag{6b}$$

$$u_z(r, \theta, 0^+) = \Omega r \cos\theta; \; b \leq r \leq a \tag{6c}$$

$$\sigma_{zz}(r, \theta, 0^+) = 0; \; a < r < \infty \tag{6d}$$

$$\sigma_{zz}(r, \theta, 0^+) = 0; \; 0 < r < b. \tag{6e}$$

(iii) *For the rigid body rotation about the z-axis*

$$u_\theta(r, \theta, 0^+) = \omega r; \; b \leq r \leq a \tag{7a}$$

$$\sigma_{\theta z}(r, \theta, 0^+) = 0; \; a < r < \infty \tag{7b}$$

$$\sigma_{\theta z}(r, \theta, 0^+) = 0; \; 0 < r < b. \tag{7c}$$

(iv) *For the rigid body translation along the x-direction*

$$u_z(r, \theta, 0^+) = 0; \; r \geq 0 \tag{8a}$$

$$u_r(r, \theta, 0^+) = \delta \cos\theta; \; b \leq r \leq a \tag{8b}$$

$$u_\theta(r, \theta, 0^+) = -\delta \sin\theta; \; b \leq r \leq a \tag{8c}$$

$$\sigma_{rz} \sin\theta + \sigma_{\theta z} \cos\theta = 0; \; r \geq 0 \tag{8d}$$

$$\sigma_{rz} \cos\theta - \sigma_{\theta z} \sin\theta = 0; \; a < r < \infty \tag{8e}$$

$$\sigma_{rz} \cos\theta - \sigma_{\theta z} \sin\theta = 0; \; 0 < r < b. \tag{8f}$$

The boundary conditions (8d), (8e) and (8f) relate to the traction vectors which act on the plane $z = 0^+$ along the y and x directions, respectively.

4. TRIPLE INTEGRAL EQUATION FORMULATION

For the integral equation formulation of these problems we seek solutions of (1) which can be obtained from Hankel transform developments of the equations. The displacement and stress fields derived from the functions Φ and Ψ should tend to zero as $(r^2+z^2)^{1/2}\to\infty$. The relevant solutions for Φ and Ψ take the following forms.

(i) *For the rigid body translation in the z-direction*

$$\Phi(r,z)=\int_0^\infty \xi[A(\xi)+zB(\xi)]\,e^{-\xi z}J_0(\xi r)\,d\xi \tag{9a}$$

$$\Psi(r,z)=0. \tag{9b}$$

(ii) *For the rigid body rotation about the y-axis*

$$\Phi(r,\theta,z)=\left\{\int_0^\infty \xi[A(\xi)+zB(\xi)]\,e^{-\xi z}J_1(\xi r)\,d\xi\right\}\cos\theta \tag{10a}$$

$$\Psi(r,\theta,z)=\left\{\int_0^\infty \xi C(\xi)\,e^{-\xi z}J_1(\xi r)\,d\xi\right\}\sin\theta. \tag{10b}$$

(iii) *For the rigid body rotation about the z-axis*

$$\Phi(r,\theta,z)=\int_0^\infty \xi[A(\xi)+zB(\xi)]\,e^{-\xi z}J_1(\xi r)\,d\xi \tag{11a}$$

$$\Psi(r,\theta,z)=0. \tag{11b}$$

(iv) *For the rigid body translation along the x-direction*

$$\Phi(r,\theta,z)=\left\{\int_0^\infty \xi[A(\xi)+zB(\xi)]\,e^{-\xi z}J_1(\xi r)\,d\xi\right\}\cos\theta \tag{12a}$$

$$\Psi(r,\theta,z)=\left\{\int_0^\infty \xi C(\xi)\,e^{-\xi z}J_1(\xi r)\,d\xi\right\}\sin\theta. \tag{12b}$$

For convenience, we define the nth order Hankel operator as follows:

$$H_n[f(\xi);r]=\int_0^\infty \xi f(\xi)J_n(\xi r)\,d\xi. \tag{13}$$

Using the integral representations of the stress functions given by (9)–(12) and the expressions for the displacement and stress components given by (3a–c) and (4a–f) it can be shown that the mixed boundary conditions (5)–(8) reduce to sets of triple integral equations for an unknown function $R_n(\xi)(n=1,2,3,4)$.

(i) For the rigid body displacement of the annular disc inclusion in the z-direction we have

$$H_0[R_1(\xi);r]=0;\ 0<r<b \tag{14a}$$

$$H_0[\xi^{-1}R_1(\xi);r]=-\frac{2\delta(1-\nu)}{(3-4\nu)};\ b\le r\le a \tag{14b}$$

$$H_0[R_1(\xi);r]=0;\ a<r<\infty. \tag{14c}$$

(ii) For the rigid rotation of the annular disc inclusion about the y-axis we have

$$H_1[\xi^{-1}R_2(\xi); r] = 0; 0 < r < b \tag{15a}$$

$$H_1[R_2(\xi); r] = -\frac{2\Omega r(1-\nu)}{(3-4\nu)}; b \le r \le a \tag{15b}$$

$$H_1[\xi^{-1}R_2(\xi); r] = 0; a < r < \infty. \tag{15c}$$

(iii) For the rigid rotation of the annular disc inclusion about the z-axis we have

$$H_1[R_3(\xi); r] = 0; 0 < r < b \tag{16a}$$

$$H_1[\xi^{-1}R_3(\xi); r] = \omega r; b \le r \le a \tag{16b}$$

$$H_1[R_3(\xi); r] = 0; a < r < \infty. \tag{16c}$$

(iv) For the lateral translation of the annular disc inclusion along the x-direction we have

$$H_1[R_4(\xi); r] = 0; 0 < r < b \tag{17a}$$

$$H_1[\xi^{-1}R_4(\xi); r] = -\frac{4\Delta(1-\nu)}{(7-8\nu)}; b \le r \le a \tag{17b}$$

$$H_1[R_4(\xi); r] = 0; a < r < \infty. \tag{17c}$$

The sets of triple integral equations defined by (14)–(17) can be solved by employing a variety of approximate techniques. Detailed expositions of these methods are given by Williams[20], Cooke[26], Tranter[27], Collins[28] and Jain and Kanwal[29]. Complete accounts of these methods are also given by Sneddon[30] and Kanwal[31]. In the present paper we shall employ the method of solution proposed by Williams[20]. In its general form, the triple system can be written as

$$H_n[R(\xi); r] = 0; 0 < r < b \tag{18}$$

$$H_n[\xi^{-1}R(\xi); r] = f(r); b \le r \le a \tag{19}$$

$$H_n[R(\xi); r] = 0; a < r < \infty. \tag{20}$$

We assume that the function $R(\xi)$ can be written in the form

$$H_n[R(\xi); r] = g(r); b < r < a. \tag{21}$$

From the Hankel inversion theorem we have

$$R(\xi) = \int_a^b rg(r)J_n(\xi r)\,\mathrm{d}r. \tag{22}$$

Using this result in (19) we obtain

$$\int_a^b ug(u)K_0(u, r)\,\mathrm{d}u = f(r); b \le r \le a \tag{23}$$

where

$$K_0(u, r) = u\int_0^\infty J_n(\xi r)J_n(\xi u)\,\mathrm{d}\xi. \tag{24}$$

We define the functions $g_1(u)$ and $g_2(u)$ such that

$$g_1(u)+g_2(u)=\begin{cases}0 & ;0\le r<b\\ g(u) & ;b\le r\le a\\ 0 & ;a<r<\infty\end{cases} \tag{25}$$

and assume that $f(r)$ admits expansions of the form

$$f_1(r)=\sum_{n=-\infty}^{\infty} a_n r^n ;0<r<a \tag{26}$$

$$f_2(r)=\sum_{n=-\infty}^{-1} a_n r^n ;b<r<\infty. \tag{27}$$

From the representations (24)–(27) it follows that the integral equation (23) reduces to two integral equations

$$\int_0^\infty uK_0(u,r)g_1(u)\,\mathrm{d}u=f_1(r);0<r<a \tag{28}$$

$$\int_0^\infty uK_0(u,r)g_2(u)\,\mathrm{d}u=f_2(r);b<r<\infty. \tag{29}$$

By making use of the identities (A1)–(A3) given in the Appendix A it can be shown that

$$\int_t^\infty tK_0(r,t)g(t)\,\mathrm{d}t=4r^{-n}\int_0^r \frac{s^{2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_s^\infty \frac{t^{1-n}g(t)\,\mathrm{d}t}{(t^2-s^2)^{1/2}};0<r<\infty \tag{30}$$

$$\int_0^\infty tK_0(r,t)g(t)\,\mathrm{d}t=4r^{n}\int_r^\infty \frac{s^{-2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_0^s \frac{t^{1+n}g(t)\,\mathrm{d}t}{(s^2-t^2)^{1/2}};0<r<\infty. \tag{31}$$

Using these results, the integral equations (28) and (29) can be expressed in the form

$$4r^{-n}\int_s^r \frac{s^{2n}\,\mathrm{d}s}{(r^2-s^2)^{1/2}}\int_s^\infty \frac{t^{1-n}g_1(t)\,\mathrm{d}t}{(t^2-s^2)^{1/2}}=f_1(r);0<r<a \tag{32}$$

$$4r^{n}\int_r^\infty \frac{s^{-2n}\,\mathrm{d}s}{(s^2-r^2)^{1/2}}\int_0^s \frac{t^{1+n}g_2(t)\,\mathrm{d}t}{(s^2-t^2)^{1/2}}=f_2(r);b<r<\infty. \tag{33}$$

The next step is to define unknown functions $S_i(r)$, $T_i(r)$ and $C_i(r)$ $(i=1,2)$ such that

$$r^n\int_r^\infty \frac{t^{1-n}g_1(t)\,\mathrm{d}t}{(t^2-r^2)^{1/2}}=\begin{cases}S_1(r);0<r<a\\ -T_1(r);a<r<\infty\end{cases} \tag{34}$$

$$r^{-n}\int_0^r \frac{t^{1+n}g_2(t)\,\mathrm{d}t}{(r^2-t^2)^{1/2}}=\begin{cases}-T_2(r);0<r<b\\ S_2(r);b<r<\infty\end{cases} \tag{35}$$

$$4r^{-n}\int_0^r \frac{s^nC_1(s)\,\mathrm{d}s}{(r^2-s^2)^{1/2}}=f_1(r);0<r<a \tag{36}$$

$$4r^{n}\int_r^\infty \frac{s^{-n}C_2(s)\,\mathrm{d}s}{(s^2-r^2)^{1/2}}=f_2(r);b<r<\infty \tag{37}$$

$$S_i(r)=C_i(r). \tag{38}$$

The four integral equations (35)–(38) can be inverted to give

$$g_1(t) = -\frac{2}{\pi} t^{n-1} \frac{d}{dt}\left[\int_t^a \frac{u^{1-n}S_1(u)\,du}{(u^2-t^2)^{1/2}} - \int_a^\infty \frac{u^{1-n}T_1(u)\,du}{(u^2-t^2)^{1/2}}\right] \quad (39)$$

$$g_2(t) = \frac{2}{\pi} t^{-n-1} \frac{d}{dt}\left[-\int_0^b \frac{u^{n+1}T_2(u)\,du}{(t^2-u^2)^{1/2}} + \int_b^t \frac{u^{n+1}S_2(u)\,du}{(t^2-u^2)^{1/2}}\right] \quad (40)$$

$$C_1(r) = \frac{1}{2\pi r^n}\frac{d}{dr}\int_0^r \frac{u^{n+1}f_1(u)\,du}{(r^2-u^2)^{1/2}};\, 0<r<a \quad (41)$$

$$C_2(r) = -\frac{r^n}{2\pi}\frac{d}{dr}\int_r^\infty \frac{u^{1-n}f_2(u)\,du}{(u^2-r^2)^{1/2}};\, b<r<\infty. \quad (42)$$

Substituting the values of $g_1(t)$ and $g_2(t)$ given by (39) and (40) into $(34)_2$ and $(35)_1$ (the subscripts denoting the second and first equations respectively) we obtain two simultaneous Fredholm integral equations of the second kind which take the forms

$$T_1(r) = l_1(r) + \frac{n!}{r^n\sqrt{(\pi\Gamma)}(n+(3/2))}\int_0^b \frac{u^{n+1}T_2(u)\,{}_2F_1((1/2), n; n+(3/2); (u^2/r^2))\,du}{(r^2-u^2)} \quad ; a<r<\infty \quad (43)$$

$$T_2(r) = l_2(r) + \frac{r^{n+1}n!}{\sqrt{(\pi\Gamma)}(n+(3/2))}\int_a^\infty \frac{u^{-n}T_1(u)\,{}_2F_1((1/2), n; n+(3/2); (r^2/u^2))\,du}{(u^2-r^2)} \quad ; 0<r<b \quad (44)$$

where ${}_2F_1$ is a hypergeometric function and $l_1(r)$ and $l_2(r)$ are given by

$$l_1(r) = -\frac{2}{\pi r^n}\int_0^r \frac{t^{2n}\,dt}{(r^2-t^2)^{1/2}}\frac{d}{dt}\int_t^a \frac{u^{1-n}S_1(u)\,du}{(u^2-t^2)^{1/2}} \quad (45)$$

$$l_2(r) = \frac{2r^n}{\pi}\int_r^\infty \frac{t^{-2n}\,dt}{(t^2-r^2)^{1/2}}\frac{d}{dt}\int_b^t \frac{u^{n+1}S_2(u)\,du}{(t^2-u^2)^{1/2}} \quad (46)$$

The integral equations (43) and (44) can be solved, by using iterative techniques, to yield expressions for $T_i(r)$; these in turn can be used in (39) and (40) to generate the expressions for $g_i(t)$. Specific results derived from the method are outlined below.

(i) For example, for the rigid body displacement of the disc inclusion in the axial direction we have

$$n=0;\, f(r) = A = \text{const.};\, f_1(r) = A;\, f_2(r) = 0. \quad (47)$$

From (38), (41) and (42) we have

$$C_1 = S_1 = \frac{A}{2\pi};\, C_2 = S_2 = 0. \quad (48)$$

Making use of (43)–(46) we find that

$$l_1(br) = \frac{1}{\pi^2}\left[\lambda r + \frac{\lambda^3 r^3}{3} + \frac{\lambda^5 r^5}{5} + \frac{\lambda^7 r^7}{7} + 0(\lambda^9)\right]$$

$$l_2(ar) = 0 \quad (49)$$

$\lambda = b/a$ and $0(\lambda^n)$ is the Landau symbol.

By iteration we obtain from (43) and (44) the following expressions for $T_i(r)$:

$$T_1(ar) = \frac{2\lambda^2}{\pi^2}\left[\frac{1}{r^2}\left(\frac{\lambda}{3} + \frac{\lambda^3}{15} + \frac{4\lambda^4}{27\pi^2} + \frac{\lambda^5}{35} + \frac{92\lambda^6}{675\pi^2}\right) + \frac{1}{r^4}\left(\frac{\lambda^3}{5} + \frac{\lambda^5}{21} + \frac{4\lambda^6}{45\pi^2}\right) + \frac{\lambda^5}{7\pi^6} + 0(\lambda^7)\right];\, 1<r<\infty \quad (50)$$

$$T_2(br) = \frac{1}{\pi^2}\left[\lambda r + \frac{\lambda^3 r^3}{3} + \frac{\lambda^5 r^5}{5} + \frac{\lambda^7 r^7}{7} + \frac{4}{\pi^2}\left\{\left(\frac{\lambda^4}{9} + \frac{14\lambda^6}{225} + \frac{4\lambda^7}{81\pi^2} + \frac{29\lambda^8}{735}\right)r\right.\right.$$

$$\left.\left. + \left(\frac{\lambda^6}{15} + \frac{22\lambda^8}{525}\right)r^3 + \frac{\lambda^2 r^5}{21}\right\} + 0(\lambda^9)\right]; 0 < r < 1. \tag{51}$$

These results can be used to develop the relevant expression for $g(t)(= g_1(t) + g_2(t))$.

(ii) For the rigid rotation of the disc inclusion about the y-axis $n = 1$; $f(r) = Br$; $f_1(r) = B$; $f_2(r) = 0$

where B is a constant

$$C_1(r) = S_1(r) = 2Br; C_2(r) = S_2(r) = 0.$$

The corresponding expressions for $T_1(ar)$ and $T_2(br)$ take the forms:

$$T_1(ar) = \frac{32Ba\lambda^5}{45\pi^2}\left[\frac{1}{r^3}\left(1 + \frac{2\lambda^2}{7}\right) + \frac{6\lambda^2}{7r^5} + 0(\lambda^4)\right]; 1 < r < \infty \tag{52}$$

$$T_2(br) = \frac{8Ba\lambda^5}{3\pi}\left[\lambda^2 r^2 + \frac{2\lambda^4 r^4}{5} + 0(\lambda^6)\right]; 0 < r < 1. \tag{53}$$

Similar results can be derived for the problems which relate to rotation of the disc inclusion about the z-axis and the translation of the inclusion along the x-axis.

5. LOAD-DISPLACEMENT RELATIONSHIPS FOR THE DISC INCLUSION

In this section we shall restrict our attention to the determination of the load–displacement relationships for the embedded disc inclusion. Such results are of importance to engineering applications.

(i) *Rigid body translation in the z-direction*

Referring to Fig. 1, we note that the eccentrically applied load P can be visualized as a combination of a "resultant" force P and a "resultant" moment $M_0 = Pc$ which acts about the y-axis. Considering the axisymmetric problem we have

$$P = 2\pi \int_b^a r[\sigma_{zz}(r, 0^-) - \sigma_{zz}(r, 0^+)]\,dr \tag{54}$$

where $\sigma_{zz}(r, 0^+)$ and $\sigma_{zz}(r, 0^-)$ refer to the normal interface stresses which act on the faces of the disc inclusion which are in contact with the halfspace regions $z > 0$ and $z < 0$ respectively. Owing to the asymmetry of the deformation in the infinite space region $\sigma_{zz}(r, 0^+) = -\sigma_{zz}(r, 0^-)$. Using the integral expression for σ_{zz}, (54) reduces to

$$P = -8\pi G \int_b^a rg(r)\,dr. \tag{55}$$

Considering (47) we can set $A = -2\delta(1-\nu)/(3-4\nu)$; consequently (55) yields

$$P = \frac{64(1-\nu)Ga\delta}{(3-4\nu)}\left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{9\lambda^5}{15\pi^2} - \frac{16\lambda^6}{27\pi^4} - \frac{92\lambda^7}{315\pi^2} - \frac{448\lambda^8}{675\pi^4} + 0(\lambda^9)\right]. \tag{56}$$

We note that as $\lambda \to 0$, (56) reduces to the classical result for the solid disc inclusion derived by Collins[1], Kanwal and Sharma[19] and Selvadurai[18].

(ii) *Rigid body rotation about the y-axis*

The resultant moment $M_0 = Pc$ is given by

$$M_0 = \pi \int_b^a r^2[\sigma_{zz}(r, \theta, 0^-) - \sigma_{zz}(r, \theta, 0^+)]\, dr. \tag{57}$$

Again $\sigma_{zz}(r, \theta, 0^+) = -\sigma_{zz}(r, \theta, 0^-)$ and

$$M_0 = -4\pi G \int_b^a r^2 g(r)\, dr. \tag{58}$$

We set the arbitrary constant (in (52) and (53)) $B = -2(1-\nu)\Omega/(3-4\nu)$ and obtain from (58) the following expression for the moment-rotation relationship:

$$M_0 = \frac{64(1-\nu)G\Omega a^3}{3(3-4\nu)}\left[1 - \frac{16\lambda^5}{15\pi^2} - \frac{64\lambda^7}{105\pi^2} + 0(\lambda^9)\right]. \tag{59}$$

Again as $\lambda \to 0$, (59) reduces to the result for the solid inclusion given by Selvadurai[7].

(iii) *Rigid body rotation about the z-axis*

The forces R act in the plane of the disc inclusion. These forces are equivalent to a resultant torque $T(=2Rd)$ which acts about the z-axis. The magnitude of T is given by

$$T = 2\pi \int_b^a r^2[\sigma_{\theta z}(r, 0^-) + \sigma_{\theta z}(r, 0^+)]\, dr \tag{60}$$

Using the results derived in the previous sections it can be shown that

$$T = \frac{32Ga^3\omega}{3}\left[1 - \frac{16\lambda^5}{15\pi^2} - \frac{64\lambda^7}{105\pi^2} + 0(\lambda^9)\right]. \tag{61}$$

The result (61) is in agreement with analogous results derived by Collins[28] for the Reissner–Sagoci problem for an annular punch.

(iv) *Rigid body translation along the x-axis*

The application of the force Q causes a rigid body translation (Δ) of the annular disc inclusion along the x-direction.

$$Q = \int_b^a \int_0^{2\pi} [T_x(r, \theta, 0^+) + T_x(r, \theta, 0^-)] r\, dr\, d\theta. \tag{62}$$

The load–displacement relationship takes the form

$$Q = \frac{64(1-\nu)Ga\Delta}{(7-8\nu)}\left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{9\lambda^5}{15\pi^2} - \frac{16\lambda^6}{27\pi^4} - \frac{92\lambda^7}{315\pi^2} - \frac{448\lambda^8}{675\pi^4} + 0(\lambda^9)\right]. \tag{63}$$

As $\lambda \to 0$, (63) reduces to the results given by Keer[2], Kassir and Sih[3] and Selvadurai[9] for the lateral translation of the embedded solid disc inclusion.

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APPENDIX A

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{J_{n-1/2}(ps) s^{n+(1/2)}\, ds}{(r^2 - s^2)^{1/2}} \tag{A1}$$

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} r^n \int_r^\infty \frac{J_{n+(1/2)}(ps) s^{-n+(1/2)}\, ds}{(s^2 - r^2)^{1/2}} \tag{A2}$$

$$\int_0^\infty p J_{n+(1/2)}(ps) J_{n+(1/2)}(pt)\, dp = \frac{\delta^*(s-t)}{(st)^{1/2}} \tag{A3}$$

where δ^* is the Dirac delta function.